

# On the inflationary flow equations

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I explore properties of the inflationary flow equations. I show that the flow equations do *not* correspond directly to inflationary dynamics. Nevertheless, they can be used as a rather complicated algorithm for generating inflationary models. I demonstrate that the flow equations can be solved analytically and give a closed form solution for the potentials to which flow equation solutions correspond. I end by considering some simpler algorithms for generating stochastic sets of slow-roll inflationary models for confrontation with observational data.

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## I. INTRODUCTION

The inflationary flow equations were introduced by Hoffman and Turner [1], and have been proposed by Kinney [2] as a way of generating large numbers of slow-roll inflation models which can be compared to observational data. They rely on defining a set of functions, the slow-roll parameters, based on derivatives of the Hubble parameter during inflation, and deriving a set of equations for their variation with time. Integration of these equations yields a trajectory in slow-roll parameter space, which can be interpreted as a variation with scale of the scalar and tensor spectra, usually written in terms of quantities such as the tensor-to-scalar ratio and the perturbation spectral indices and their running.

Although these equations have been employed to make comparisons with observations [3, 4, 5], as yet no clear connection has been made between the inflationary dynamics and the flow equations. As I will explain in this paper, there is in fact *no* direct connection between these two; the flow equations do not encode any physical model of inflationary dynamics. Despite this, it turns out that they can be used to generate inflationary models, though they represent quite a complicated algorithm for doing so. As it happens, this can be highlighted for slow-roll inflation by obtaining a closed form analytical solution to the flow equations and their relation to the inflationary potential.

## II. THE FLOW EQUATIONS AND THEIR RELATION TO INFLATION

### A. The flow equations

In this section I follow closely the notation and presentation by Kinney [2]. We take as our fundamental quantity the evolution of the Hubble parameter  $H$  as a function of  $\phi$  (often called the Hamilton–Jacobi approach to inflation [6]). From this we define a set of Hubble slow-

roll parameters,<sup>1</sup>

$$\epsilon(\phi) \equiv \frac{m_{\text{Pl}}^2}{4\pi} \left( \frac{H'(\phi)}{H(\phi)} \right)^2 ; \quad (1)$$

$${}^\ell \lambda_H \equiv \left( \frac{m_{\text{Pl}}^2}{4\pi} \right)^\ell \frac{(H')^{\ell-1}}{H^\ell} \frac{d^{(\ell+1)}H}{d\phi^{(\ell+1)}} ; \quad \ell \geq 1 \quad (2)$$

where primes are derivatives with respect to the scalar field. For example, the parameter  ${}^1 \lambda_H$  equals  $(m_{\text{Pl}}^2/4\pi) H''/H$  and is often denoted  $\eta(\phi)$ .

As the successive parameters feature an ever high number of derivatives, we can construct a hierarchy of flow equations, with the derivative of  $\epsilon$  given in terms of parameters up to  ${}^1 \lambda_H$ , the derivative of  ${}^1 \lambda_H$  in terms of parameters up to  ${}^2 \lambda_H$  etc. Rather than the derivative with respect to  $\phi$ , it is convenient to take the derivative with respect to the number of  $e$ -foldings of inflation  $N$ , using the relation

$$\frac{d}{dN} = \frac{m_{\text{Pl}}}{2\sqrt{\pi}} \sqrt{\epsilon} \frac{d}{d\phi} . \quad (3)$$

Kinney derives the flow equations, using a convenient definition  $\sigma \equiv 2({}^1 \lambda_H) - 4\epsilon$ , as

$$\begin{aligned} \frac{d\epsilon}{dN} &= \epsilon(\sigma + 2\epsilon) ; \\ \frac{d\sigma}{dN} &= -5\epsilon\sigma - 12\epsilon^2 + 2({}^2 \lambda_H) ; \\ \frac{d({}^\ell \lambda_H)}{dN} &= \left[ \frac{\ell-1}{2} \sigma + (\ell-2)\epsilon \right] ({}^\ell \lambda_H) + {}^{\ell+1} \lambda_H ; \quad \ell \geq 2 . \end{aligned} \quad (4)$$

In order to solve this infinite series, it must be truncated by setting a sufficiently high slow-roll parameter to zero, i.e.  ${}^{M+1} \lambda_H = 0$  for some suitably large  $M$  such as  $M = 5$  [2]. Eqs. (4) then comprise a closed set, and can be integrated by choosing initial conditions for the

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<sup>1</sup> This approach was originally suggested by Liddle, Parsons and Barrow [7], but here I follow the notation of Kinney [2] rather than that paper.

parameters  $\epsilon$ ,  ${}^1\lambda_H$ , ...,  ${}^M\lambda_H$ . Typically either at some point  $\epsilon$  reaches one, indicating the end of inflation, or the model reaches a late-time attractor with perpetual inflation.

## B. Interpretation of the flow equations

The main purpose of this paper concerns understanding the dynamical properties of the flow equations and how they relate to the inflationary dynamics. The answer is that they do not at all, because *the above discussion has been carried out without ever mentioning the main dynamical equation of inflation*. The missing equation is the Hamilton–Jacobi equation (equivalent to the Friedmann equation)

$$[H'(\phi)]^2 - \frac{12\pi}{m_{Pl}^2} H^2(\phi) = -\frac{32\pi^2}{m_{Pl}^4} V(\phi), \quad (5)$$

which tells us how the expansion rate is linked to the potential  $V(\phi)$  for the inflaton, which has also not been mentioned at all up to this point.

In fact, had the flow equations been written using  $d/d\phi$ , they would have amounted to a trivial set of relations amongst derivatives of the Hubble parameter; substituting in the definitions in terms of  $H(\phi)$  would lead to a set of tautologous equations. The situation is made less trivial by the use of  $N$  rather than  $\phi$ , which does require some dynamical information from the equations of motion; however either of these parameters is just measuring the distance along the trajectory, and does nothing to alter the actual trajectories in parameter space, which is what the Hamilton–Jacobi equation ought to be determining. To summarize, solving the flow equations Eqs. (4) has nothing to do with solving the inflationary equations of motion.

Why is it, then, that the flow equations do seem to correspond to inflationary models (e.g. Refs. [4, 5, 8])? The reason is that the ultimate output of the flow equations is a function  $\epsilon(\phi)$  (the evolution of any other slow-roll parameter could be derived from this). Long ago, I wrote a paper [9] which introduced the idea that a general slow-roll inflation model could be specified by giving the function  $\epsilon(\phi)$ , which should be less than unity for inflation to take place. This is in contrast to the more traditional view that a model is specified by  $V(\phi)$ , or perhaps  $H(\phi)$  [6, 10], but there is a mapping between them: given  $\epsilon(\phi)$  we can then obtain

$$H(\phi) = H_i \exp \left( \int_{\phi_i}^{\phi} \sqrt{4\pi\epsilon(\phi)} \frac{d\phi}{m_{Pl}} \right), \quad (6)$$

from Eq. (1) and

$$V(\phi) = \frac{3m_{Pl}^2}{8\pi} H^2(\phi) \left[ 1 - \frac{1}{3}\epsilon(\phi) \right], \quad (7)$$

from Eq. (5). Accordingly, for any function  $\epsilon(\phi)$ , which must be between zero and one, one can obtain a slow-roll inflation model with potential  $V(\phi)$  which gives that  $\epsilon(\phi)$ . Indeed, Easter and Kinney [8] used a version of these relations to numerically obtain potentials from the flow equations.

In this light, we therefore see that what the flow equations represent is simply a rather complicated algorithm for generating functions  $\epsilon(\phi)$ , which have the correct general form to be interpreted as inflationary models. In themselves, they do not incorporate the inflationary dynamics.<sup>2</sup>

That the flow equations do not incorporate the inflationary dynamics directly raises an interesting point — that the outcome of the flow equation analysis for observable quantities would be largely unchanged even if the dynamical equations were different. For example, one might consider the possibility that the correct dynamical equations are those of the simplest braneworld model, based on the Type II Randall–Sundrum model [11], where the Friedmann equation is

$$H^2 = \frac{8\pi}{3m_{Pl}^2} \rho \left( 1 + \frac{\rho}{2\lambda} \right), \quad (8)$$

where  $\lambda$  is the brane tension. There would be a slight change to the flow equations, as the equation relating  $N$  and  $\phi$  is now changed [12]; this does not change the trajectories themselves, but it does change the measure of length along them, and in particular would alter the point corresponding to 60 e-foldings. However the fundamental structure of the flow equations, and the trajectories corresponding to their solutions, are unchanged despite the change in the underlying dynamical assumption.

## III. ANALYTIC SOLUTION OF THE FLOW EQUATIONS

Having concluded that the flow equations represent an algorithm for generating suitable functions  $\epsilon(\phi)$ , it is possible to discover the models that this procedure corresponds to, where from now on I will consider only the standard cosmology of Eq. (5). As far as I can judge, the papers in the literature so far solve the truncated flow equations numerically [1, 2, 8], but in fact it is possible to solve them analytically as follows.

The truncation requires that  ${}^{M+1}\lambda_H$  equals zero at all times. However if we look at the definition of that parameter, Eq. (2), we see that this is equivalent to the

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<sup>2</sup> This conclusion is somewhat less apparent for the flow equations written in terms of observables by Hoffman and Turner [1], but Kinney [2] showed that their equations are identical to the flow equations he discussed.

statement that

$$\frac{d^{M+2}H}{d\phi^{M+2}} = 0, \quad (9)$$

for all  $\phi$ . Hence  $H(\phi)$  is a polynomial of order  $M + 1$ , which we can write as

$$H(\phi) = H_0 (1 + A_1\phi + \cdots + A_{M+1}\phi^{M+1}) \quad (10)$$

where the  $A_i$  are constants. Further, without loss of generality we can choose the initial value of  $\phi$  (from which the flow equation integration begins) to be equal to zero.

Then the constants in Eq. (10) can be easily related to the initial values of the slow-roll parameters for the flow equation integration, for example

$$\epsilon(\phi) = \frac{m_{\text{Pl}}^2}{4\pi} \left( \frac{A_1 + \cdots + (M+1)A_{M+1}\phi^M}{1 + A_1\phi + \cdots + A_{M+1}\phi^{M+1}} \right)^2 \quad (11)$$

and so

$$\epsilon(\phi = 0) = \frac{A_1^2 m_{\text{Pl}}^2}{4\pi}, \quad (12)$$

with the sign of  $A_1$  determining which direction the field rolls. Similarly, the initial values of  $\ell\lambda_H$  can be related to  $A_{\ell+1}$ . The constant  $H_0$  is not fixed by the dynamical equations, but potentially can be determined observationally by observing the amplitude of tensor perturbations.

Having this closed form solution for  $H(\phi)$ , we can use Eq. (7) to determine the equivalent potential, which is

$$V(\phi) = \frac{3m_{\text{Pl}}^2}{8\pi} H_0^2 (1 + A_1\phi + \cdots + A_{M+1}\phi^{M+1})^2 \quad (13)$$

$$\times \left[ 1 - \frac{1}{3} \frac{m_{\text{Pl}}^2}{4\pi} \left( \frac{A_1 + \cdots + (M+1)A_{M+1}\phi^M}{1 + A_1\phi + \cdots + A_{M+1}\phi^{M+1}} \right)^2 \right].$$

As there are so many undetermined constants, this potential can represent a wide range of possible behaviours. However these can be divided into two main classes; either the potential becomes negative, in which case slow-roll will fail and inflation end sometime before the potential reaches zero, or the potential develops a minimum at a positive value, in which case the field asymptotes there driving eternal inflation. These two possibilities represent the two late-time behaviours seen in solutions of the flow equations, with this attractor structure holding even if the flow equation hierarchy is taken to infinite order [2].

In conclusion, the flow equation approach to inflation is equivalent to considering the set of models described by Eq. (13), where the constant  $A_1$  is to be chosen to be consistent with the condition that inflation is occurring initially, i.e.  $|A_1|m_{\text{Pl}} < \sqrt{4\pi}$ , and the other constants are ordinarily to be chosen to satisfy the slow-roll conditions  $|\ell\lambda_H| \ll 1$ .

#### IV. ALTERNATIVE APPROACHES TO STOCHASTIC MODEL-BUILDING

The previous discussion indicates that one need not employ the flow equation formalism in order to build up a stochastic set of inflation models by randomly drawing slow-roll parameters. Several approaches suggest themselves.

Firstly, the results from the flow equations approach can be reproduced by working directly with the parametrized potential of Eq. (13). Models which are able to match present observations (see e.g. Refs. [4, 13]) lie close to the extreme slow-roll limit, so it is probably sufficient to analyze them using the slow-roll approximation, though by construction they have an exact analytical solution to the equations of motion given by Eq. (10) and so this is unnecessary.

More generally, one can ask whether there is any real benefit in using the flow equations to generate the function  $\epsilon(\phi)$ , given that they have nothing to do with inflationary dynamics. Working with Eq. (13) is therefore unlikely to be any more reasonable than using simply Taylor expanding  $V(\phi)$  itself, and then solving either using slow-roll or numerically.

As we have seen, the flow equations approach is in fact equivalent to Taylor expanding  $H(\phi)$ , Eq. (10). However yet another alternative, perhaps the most appealing of all, is to use an expansion to generate  $\epsilon(\phi)$  as the fundamental quantity, but to do so in a more direct way than the flow equations algorithm. Because  $\epsilon(\phi)$  directly tells us whether inflation is indeed taking place, it seems an attractive starting point, though the need to keep it positive does restrict the allowed expansions. While this paper aims to elucidate the nature of the flow equations approach, it would be interesting to make a direct comparison of some of the methods outlined in this section to contrast with the flow equations. That ought to give some indication of whether the ‘preference’ of flow-equation models for certain regions of observable parameter space [1, 2] is a robust prediction, or a hidden consequence of the particular method.

#### V. CONCLUSIONS

In this paper I have shown that the flow equations approach does not directly incorporate inflationary dynamics. Rather, it represents a complicated algorithm for generating functions  $\epsilon(\phi)$ , which can then be used to generate inflationary models with dynamics matching those of the flow equations. I have shown that one can analytically determine the set of inflationary potentials which correspond to solutions of the truncated flow equations. The generality with which the flow equations treat inflationary dynamics therefore depends on the extent to which the family of potentials given by Eq. (13) may have enough free parameters to be able to represent broad classes of possible potential shapes. However, the

fact that the flow equations are so loosely related to the inflationary dynamics must cast some doubt on conclusions drawn from them on how densely inflation models sample different regions of observable parameter space.

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